# Exam 2 Solutions.

1. Use Simpson's rule with $n = 4$ to estimate

$$\ln 25 = \int_1^5 \frac{5}{x} dx.$$

**Solution:** Here we have $a = 1$, $b = 5$, $\Delta x = \frac{5-1}{4} = 1$, and $f(x) = \frac{5}{x}$. Therefore, using the formula for Simpson's rule, we have

$$\int_1^5 \frac{5}{x}\, dx \approx \frac{1}{3}\left[\frac{5}{1} + 4\frac{5}{2} + 2\frac{5}{3} + 4\frac{5}{4} + \frac{5}{5}\right] = \frac{1}{3}\left[\frac{5}{1} + \frac{20}{2} + \frac{10}{3} + \frac{20}{4} + \frac{5}{5}\right].$$

2. Evaluate the improper integral

$$\int_1^4 \frac{1}{(x-2)^3} dx.$$

**Solution:** We have

$$\int_1^4 \frac{1}{(x-2)^3} dx = \int_1^2 \frac{1}{(x-2)^3} dx + \int_2^4 \frac{1}{(x-2)^3} dx.$$

Now

$$\begin{aligned}
\int_1^2 \frac{1}{(x-2)^3} dx &= \lim_{t \to 2^-} \int_1^t \frac{1}{(x-2)^3} dx \\
&= \lim_{t \to 2^-} \left. \frac{-1}{2(x-2)^2} \right|_1^t \\
&= \lim_{t \to 2^-} \left( \frac{-1}{2(t-2)^2} - \frac{-1}{2(1-2)^2} \right) \\
&= -\infty.
\end{aligned}$$

Since one of the two integrals is divergent, the original integral diverges. [In fact, in this case, $\int_2^4 \frac{1}{(x-2)^3}\, dx$ diverges as well, but we only need one of these two improper integrals here to diverge to conclude that the original integral diverges.]

3. Evaluate the improper integral

$$\int_1^\infty \frac{1}{\sqrt{x}(1+x)} dx.$$

**Solution:** We'll need an antiderivative of our function, so we do that first. Using $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}}\, dx$ (and so $x = u^2$), we have

$$\int \frac{1}{\sqrt{x}(1+x)} dx = 2 \int \frac{1}{1+u^2} du = 2 \arctan u + C = 2 \arctan \sqrt{x} + C.$$

Therefore

$$\begin{aligned}
\int_1^\infty \frac{1}{\sqrt{x}(1+x)}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}(1+x)}dx \\
&= \lim_{t\to\infty} 2\arctan\sqrt{x}\Big|_1^t \\
&= \lim_{t\to\infty} 2\left(\arctan\sqrt{t}-\arctan 1\right) \\
&= 2\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

4. Which of the following is the correct expression for the arc length of the curve

$$y = e^{\,x^2/2}$$

between the points $(1,\sqrt{e})$ and $(2,e^2)$?

**Solution:** The formula is $\displaystyle\int_1^2 \sqrt{1+(y')^2}dx$. If $y = e^{x^2/2}$ then by the chain rule $y' = xe^{x^2/2}$, and squaring this gives $x^2e^{x^2}$. Therefore the answer is

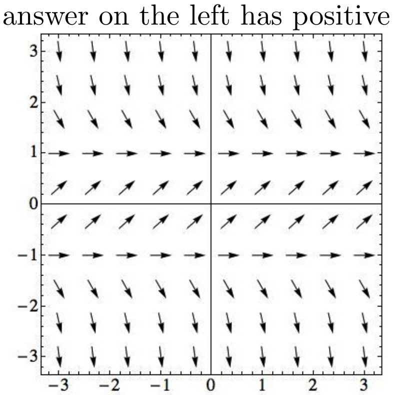
$$\int_1^2 \sqrt{1+x^2e^{x^2}}dx$$

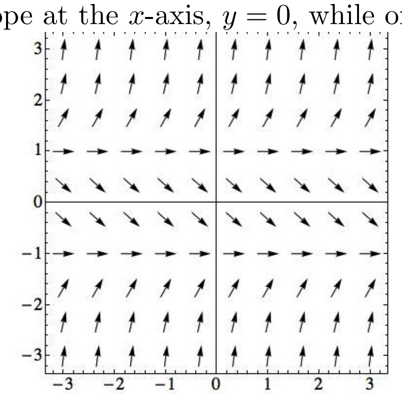
.

5. Which of the following gives the direction field for the differential equation

$$\frac{dy}{dx} = y^2 - 1\ ?$$

**Solution:** The most recognizable feature of a direction field is usually the location of its horizontal arrows, representing zero slope. This happens when $y^2-1=0$, or $y=\pm1$. Only two of the possible answers have horizontal arrows here, and they are shown below. We can see that one answer is the negative of the other. We can tell the difference between them by seeing that the answer on the left has positive slope at the $x$-axis, $y=0$, while on the right we have negative slope.




But $y^2-1=-1$ at $y=0$, so it is the option with negative slope, the second picture shown above.

6. Use Euler's method with step size 0.5 to estimate $y(1.5)$ where $y(x)$ is the solution to the initial value problem

$$y' = y^2 + 2x, \quad y(0.5) = 1.$$

**Solution:** Answer: This can be answered using the formula for Euler's method, but instead we will calculate by understanding what the formula means. This is a little slower, but is easier to understand and there is a smaller chance of making an arithmetical error. At $x = 0.5, y = 1$, the slope of the solution $y(x)$ is given by $y'(0.5) = 1^2 + 2(0.5) = 2$. If we take a step of size 0.5, and $y$ is changing with slope 2, then $y$ increases by $(0.5)(2) = 1$ to give an estimate of $y(1) \approx 1 + 1 = 2$. Now if $x = 1, y = 2$, the slope is $2^2 + 2(1) = 6$, so over a step of 0.5 $y$ increases by $(0.5)(6) = 3$ to give $y(1.5) \approx 2 + 3 = 5$.

7. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{3y}{2x+1}.$$

**Solution:** This equation is separable. Separating we find

$$\frac{1}{3}\frac{1}{y}y' = \frac{1}{2x+1},$$

multiplying by $dx$ and integrating we get

$$\frac{1}{3}\ln y dy = \frac{1}{2}\ln(2x+1) + C.$$

Multiply by 3 and exponentiate to get

$$y = e^{\frac{3}{2}\ln(2x+1)+C}.$$

Simplifying yields

$$y = C(2x+1)^{\frac{3}{2}}.$$

8. Find the solution of the differential equation

$$\frac{dy}{dx} - \left[\frac{2x}{x^2+4}\right]y = (x^2+4)\cos x$$

with initial condition $y(0) = 1$.

**Solution:** This is a first order linear equation. We define the integrating factor $I(x) = e^{\int -\left[\frac{2x}{x^2+4}\right]dx}$. Letting $u = x^2 + 4$ we find

$$I(x) = e^{-\int \frac{1}{u}dx} = \frac{1}{u} = \frac{1}{x^2+4}.$$

Multiplying by the integrating factor we get

$$\frac{d}{dx}\left[\frac{1}{x^2+4}y\right] = \frac{1}{x^2+4}(x^2+4)\cos x.$$

Integrating both sides, and applying the Fundamental Theorem of Calculus, we have

$$\frac{1}{x^2+4}y = \sin(x) + C,$$

therefore,

$$y = (x^2+4)\sin(x) + C(x^2+4).$$

Using the initial condition, $y(0) = 1$, we have $1 = (0^2+4)\sin(0) + C(0^2+4) = 4C$ which implies $C = \frac{1}{4}$. The solution is therefore, $y = (x^2+4)\sin(x) + \frac{1}{4}(x^2+4)$.

9. Determine if the sequence given by $a_n = ne^{-2n}$ converges or diverges and if it converges find

$$\lim_{n\to\infty} ne^{-2n}$$

Solution: The sequence converges to zero. We define the function $f(x) = x$ and $g(x) = e^{-2x}$ and we see that all the points of the sequence correspond to points of the function $y(x) = f(x)g(x)$. We apply L'Hospital's rule to find the limit.

Our goal is to do a $u$-substitution, with $u = \tan x$. Thus, $du = \sec^2(x)dx$ so we will leave one $\sec^2 x$ to become the $du$, and convert the other $\sec^2 x$ into $1 + \tan^2 x$. So we have:

$$\int \tan^{100} x \sec^4 x dx = \int \tan^{100} x\,(1+\tan^2 x)\sec^2(x)dx = \int u^{100}(1+u^2)du$$

We integrate, and then back substitute to get

$$\int \tan^{100} x \sec^4 x dx = \int u^{100} + u^{102} du = \frac{u^{101}}{101} + \frac{u^{103}}{103} + C = \frac{\tan^{101} x}{101} + \frac{\tan^{103} x}{103} + C$$

10. Consider the following sequences:

$$(I)\ \left\{(-1)^n \frac{n^2-1}{2n^2+1}\right\}_{n=1}^{\infty} \qquad (II)\ \left\{(-1)^n \frac{n^2-1}{e^n}\right\}_{n=1}^{\infty} \qquad (III)\ \left\{(-1)^n n\ln(n)\right\}_{n=1}^{\infty}$$

Which of these sequences converge and which diverge?

**Solution:** Notice that an alternating sequence $\{(-1)^n a_n\}$ converges if and only if $\lim_{n\to\infty} a_n = 0$, since otherwise the sequence would alternate between positive and negative numbers not approaching a single value.

Now $\lim_{n\to\infty} \frac{n^2-1}{2n^2+1} = 2$, so $\left\{(-1)^n \frac{n^2-1}{2n^2+1}\right\}_{n=1}^{\infty}$ diverges;

$\lim_{n\to\infty} \frac{n^2-1}{e^n} = 0$ (use L'Hopital's rule) hence $\left\{(-1)^n \frac{n^2-1}{e^n}\right\}_{n=1}^{\infty}$ converges;

$\lim_{n\to\infty} n\ln(n) = \infty$ hence $\left\{(-1)^n n\ln(n)\right\}_{n=1}^{\infty}$ diverges

11. Complete the following sentences using the words *converges* and *diverges* :

$\int_1^{\infty} \frac{1}{x^p}$ ________________ if $p > 1$ and ________________ if $p \leq 1$.

$\int_0^{1} \frac{1}{x^p}$ ________________ if $p > 1$ and ________________ if $p \leq 1$.

Decide whether the following improper integrals converge or diverge by comparing them to a known integral. In each case, state which integral you are comparing the given integral to and state clearly why you can conclude convergence or divergence.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^2+x+5}\,dx$

(b) $\displaystyle\int_1^{\infty} \frac{1}{xe^x}\,dx$

**Solution:**

$\int_1^{\infty} \frac{1}{x^p} dx$ <u>converges</u> if $p > 1$ and <u>diverges</u> if $p \leq 1$.

$\int_0^1 \frac{1}{x^p}$ <u>diverges</u> if $p \geq 1$ and <u>converges</u> if $p \leq 1$.

Notice that $1/(x^2+x+5) < 1/x^2$ for $x$ in $(1, \infty)$, hence $\int_1^{\infty} \frac{1}{x^2+x+5}\, dx < \int_1^{\infty} \frac{1}{x^2}\, dx$. Since the bigger one converges (by above), so does the smaller one.
Also, $1/(xe^x) < 1/e^x$ for $x$ in $(1, \infty)$, hence $\int_1^{\infty} \frac{1}{xe^x}\, dx < \int_1^{\infty} \frac{1}{e^x}\, dx = -e^{-x}\Big|_1^{\infty} = e^{-1}$. Since the bigger one converges, so does the smaller one.

12. Find the centroid of the region enclosed by the curves $y = x^2$ and $y = x^3$.

    **Solution:** Setting $x^3 = x^2$, we get that this curves intersect at $x = 0$ and $x = 1$. Since $x^2 > x^3$ for $0 \leq x \leq 1$ we get that the area of the region is $\int_0^1 (x^2 - x^3)dx = 1/3 - 1/4 = 1/12$. Hence $1/A = 12$. So, $\bar{x} = 1/A \int_0^1 x[x^2 - x^3]dx = 12(1/4 - 1/5) = 3/5$; and $\bar{y} = (1/2)(1/A) \int_0^1 (x^2)^2 - (x^3)^2 dx = 12/2(1/5 - 1/7) = 12/35$.

13. Find the family of orthogonal trajectories to the family of curves given by

$$y = k\sqrt{x}.$$

**Solution:** Our goal is to find a family of curves that are orthogonal (have perpendicular slopes) to the original family. So we first find the slopes of the given family, $\frac{dy}{dx} = \frac{k}{2\sqrt{x}}$. Before we proceed, we need to solve for $k$ in terms of $y$ and $x$. We use the original equation, $y = k\sqrt{x} \implies \frac{y}{\sqrt{x}} = k$. We plug this back into the derivative $\frac{dy}{dx} = \frac{k}{2\sqrt{x}} = \frac{\frac{y}{\sqrt{x}}}{2\sqrt{x}} = \frac{y}{2x}$

The slope of the orthogonal trajectories is the negative reciprocal of these slopes, so we get a differential equation $\frac{dy}{dx} = \frac{-2x}{y}$. This is a separable differential equation, so we separate and solve.

$$\frac{dy}{dx} = \frac{-2x}{y} \implies y\ dy = -2x\ dx \implies \frac{1}{2}y^2 = -x^2 + C \implies \frac{y^2}{2} + x^2 = C$$

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